

## Unit 5: Cubic Splines

Let  $K = \{x_0, \dots, x_m\}$  be a set of given knots with

$$a = x_0 < x_1 < \dots < x_m = b$$

**Definition. [11.2]** A function  $s \in C^2[a, b]$  is called a cubic spline on  $[a, b]$ , if  $s$  is a cubic polynomial  $s_i$  in each interval  $[x_i, x_{i+1}]$ .

It is called a cubic interpolating spline if  $s(x_i) = y_i$  for given values  $y_i$ .

## 5.1: Cubic Splines

Interpolating cubic splines need two additional conditions to be uniquely defined

**Definition. [11.3]** *An cubic interpolatory spline  $s$  is called a natural spline if*

$$s''(x_0) = s''(x_m) = 0$$

## 5.2: Cubic Splines - Construction

We construct an interpolating in a different but equivalent way than in the textbook:

Ansatz for  $m$  the piecewise polynomials

$$s_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i$$

By fixing the  $4m$  free coefficients  $a_i, b_i, c_i, d_i, i = 0 : m - 1$  the entire spline is fixed.

## 5.3: Cubic Splines-Construction

We need  $4m$  conditions to fix the coefficients

$$(1) \quad s_i(x_i) = y_i, \quad \text{for } i = 0 : m - 1,$$

$$(2) \quad s_{m-1} = y_m, \quad \text{1 condition}$$

$$(3) \quad s_i(x_{i+1}) = s_{i+1}(x_{i+1}), \quad \text{for } i = 0 : m - 2,$$

$$(4) \quad s'_i(x_{i+1}) = s'_{i+1}(x_{i+1}), \quad \text{for } i = 0 : m - 2,$$

$$(5) \quad s''_i(x_{i+1}) = s''_{i+1}(x_{i+1}), \quad \text{for } i = 0 : m - 2,$$

These are  $4m - 2$  conditions. We need two extra.

## 5.4: Cubic Splines-Boundary Conditions

We can define two extra boundary conditions. One has several alternatives:

**Natural Spline**

$$s''_0(x_0) = 0 \text{ and } s''_{m-1}(x_m) = 0$$

**End Slope Spline**

$$s'_0(x_0) = y'_0 \text{ and } s'_{m-1}(x_m) = y'_m$$

**Periodic Spline**

$$s'_0(x_0) = s'_{m-1}(x_m) \text{ and } s''_0(x_0) = s''_{m-1}(x_m)$$

**Not-a-Knot Spline**

$$s'''_0(x_1) = s'''_1(x_1) \text{ and } s'''_{m-2}(x_{m-1}) = s'''_{m-1}(x_{m-1})$$

We consider here natural splines.

MATLAB uses splines with a not-a-knot condition.

## 5.5: Natural Splines Construction

$$s_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i$$

Let  $h = x_{i+1} - x_i$  equidistant spacing

$$s'_i(t_{i+1}) = 3a_i h^2 + 2b_i h + c_i$$

$$s''_i(t_{i+1}) = 6a_i h + 2b_i$$

From Condition (1) we get  $d_i = y_i$ .

We introduce new variables for the second derivatives at  $x_i$ , i.e.

$$\sigma_i := s''(x_i) = 6a_i(x_i - x_i) + 2b_i = 2b_i \quad i = 0 : m$$

## 5.6: Natural Splines Construction

(Cont.)

Thus  $b_i = \frac{\sigma_i}{2}$ .

From

$$\sigma_{i+1} = 6a_i h + 2b_i.$$

and Condition (5) we get

$$a_i = \frac{\sigma_{i+1} - \sigma_i}{6h}.$$

By Condition (3) we get

$$y_{i+1} = a_i h^3 + b_i h^2 + c_i h + y_i.$$

## 5.7: Natural Splines Construction

(Cont.)

... and after inserting the high-lighted expressions for  $a_i$  and  $b_i$  we get

$$y_{i+1} = \left( \frac{\sigma_{i+1} - \sigma_i}{6h} \right) h^3 + \frac{\sigma_i}{2} h^2 + c_i h + y_i.$$

From that we get  $c_i$ :

$$c_i = \frac{y_{i+1} - y_i}{h} - h \frac{2\sigma_i + \sigma_{i+1}}{6}.$$

Using now Condition (4) gives a relation between  $c_i$  and  $c_{i+1}$

$$c_{i+1} = 3a_i h^2 + 2b_i h + c_i.$$

## 5.8: Natural Splines Construction

(Cont.)

Inserting now the expressions for  $a_i, b_i$  and  $c_i$ , using Condition (2) and simplifying finally gives the central recursion formula:

$$\sigma_{i-1} + 4\sigma_i + \sigma_{i+1} = 6\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right)$$

with  $i = 1, \dots, m - 1$ .

We consider now natural boundary conditions

$$\sigma_0 = \sigma_m = 0.$$

## 5.9: Natural Splines Construction

(Cont.)

Finally we rewrite this all a system of linear equations

$$\begin{pmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & 1 & \cdots & 1 & \\ & & & \cdots & 1 \\ & & & 1 & 4 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \vdots \\ \sigma_{m-1} \end{pmatrix} = \frac{6}{h^2} \begin{pmatrix} y_2 - 2y_1 + y_0 \\ y_3 - 2y_2 + y_1 \\ \vdots \\ \vdots \\ y_n - 2y_{n-1} + y_{n-2} \end{pmatrix}$$

First, this system is solved and then the coefficients  $a_i, b_i, c_i, d_i$  are determined by the high-lighted equations.